

A CHARACTERIZATION OF DOMAINS IN \mathbb{C}^n WITH LOCALLY LEVI-FLAT BOUNDARIES

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ABSTRACT. A domain in \mathbb{C}^n with Levi-flat boundary near a given point is characterized in terms of the boundary behavior of the Kobayashi or Bergman metrics, or of the Bergman kernel. Some results are given in the case of intermediate values of the rank of the Levi form.

1. THE MAIN RESULTS

This note is motivated by Ohsawa's question [13, Q2] about the characterization of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with locally Levi-flat boundary (i.e. the rank of the Levi form is zero) in terms of the boundary behavior of the Bergman metric β_Ω . This follows from some of the results of Siqi Fu's Ph. D. dissertation [6], which have been made more widely available recently [7]. We shall however give an answer to Ohsawa's question, as well as to a similar question for the Kobayashi metric κ_Ω , using weaker regularity assumptions and somewhat different methods. The Levi-flat case is only a special case of Fu's results, who also related the rank of the Levi form and rate of growth of the Bergman kernel. We provide similar results using our own methods, under smoothness hypotheses which are slightly different from Fu's.

Recall that

$$\kappa_\Omega(z; X) = \inf\{|t| : \exists \varphi \in \mathcal{O}(\mathbb{D}, \Omega) : \varphi(0) = z, t\varphi'(0) = X\}$$

(\mathbb{D} is the unit disc) and

$$\beta_\Omega(z; X) = m_\Omega(z; X)/k_\Omega(z),$$

where

$$k_\Omega(z) = \sup\{|f(z)| : f \in L^2(\Omega) \cap \mathcal{O}(\Omega), \|f\|_{L^2(\Omega)} \leq 1\}$$

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is the square root of Bergman kernel (restricted to the diagonal) and $m_\Omega(z; X) = \sup\{|f'(z)X| : f \in L^2(\Omega) \cap \mathcal{O}(\Omega), \|f\|_{L^2(\Omega)} = 1, f(z) = 0\}$.

Suppose that p is a \mathcal{C}^2 -smooth boundary point of Ω . Then for each $z \in D$ near p there exists a unique point $p(z) \in \partial\Omega$ such that $z = p(z) + \delta_\Omega(z)n_{p(z)}$, where δ_Ω is the distance ∂D and $n_{p(z)}$ is the inner normal vector at $p(z)$. For such a z and $X \in \mathbb{C}^n$, there is a unique orthogonal decomposition $X = X_N + X_T$ where $X_T \in T_{p(z)}^\mathbb{C}\partial\Omega$.

Theorem 1. *Let p be a boundary point of a bounded domain $\Omega \subset \mathbb{C}^n$ such that $\partial\Omega$ is \mathcal{C}^2 -smooth near p .*

Then $\partial\Omega$ is Levi-flat near p if and only if there exist a neighborhood U of p and a constant $c > 1$ such that for any $z \in \Omega \cap U$ and any $X \in \mathbb{C}^n \setminus \{0\}$,

$$(1) \quad c^{-1} < \frac{\kappa_\Omega(z; X)}{\frac{\|X_N\|}{\delta_\Omega(z)} + \|X\|} < c.$$

Theorem 2. *Let p be a boundary point of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\partial\Omega$ is \mathcal{C}^2 -smooth near p . Then the following three conditions are equivalent:*

- (i) $\partial\Omega$ is Levi-flat near p ;
- (ii) there exist a neighborhood U of p and a constant $c > 1$ such that for any $z \in \Omega \cap U$,

$$(2) \quad c^{-1} < k_\Omega(z)\delta_\Omega(z) < c;$$

- (iii) there exist a neighborhood U of p and a constant $c > 1$ such that for any $z \in \Omega \cap U$ and any $X \in \mathbb{C}^n \setminus \{0\}$,

$$(3) \quad c^{-1} < \frac{\beta_\Omega(z; X)}{\frac{\|X_N\|}{\delta_\Omega(z)} + \|X\|} < c.$$

The proofs are given in sections 2, 3 and 5.

Section 4 is devoted to the question of how we can recover the rank of the Levi form at a boundary point from the growth of the Bergman kernel near that point. There is a good fit in the \mathcal{C}^∞ -smooth case, see Theorem 6.

In the last section 6, refinements of the above estimates are proved in the convex and the planar cases.

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Remark. We had not realized, as we should have, that eighteen years ago, chapter IV of Fu's dissertation [6] had answered Ohsawa's question and various extensions to intermediate ranks and other metrics, in the

smooth, pseudoconvex case. Although many technical tools are the same, such as choosing a normal form of the coordinates to prove that certain polydisks (or more general sets) are contained in the domain Ω , some differences should be noted between his work and ours. We refer to the generally available version [7].

- Our Theorems 1 and 2 and Proposition 5 only require \mathcal{C}^2 smoothness, exploiting the optimal hypotheses of [12].

In [7, Theorem 1.1], which relates the rank of the Levi form (assumed to be constant in a neighborhood of the base point) and the behavior of pseudometrics, Ω is assumed to be “smooth” near the point under consideration, although after examination it seems that the crucial tool [3, Theorem 6.1] does not require more than \mathcal{C}^2 smoothness and that a \mathcal{C}^3 assumption is enough to obtain [7, Proposition 3.2] (and perhaps one could improve that proof to require only the \mathcal{C}^2 assumption).

- In [7, Theorem 1.1], Ω is assumed to be pseudoconvex (because global plurisubharmonic functions are constructed, which we dispense with, and [7] deals with the Sibony metric, which we do not treat); our Theorem 1 about the Kobayashi infinitesimal pseudometric does not require pseudoconvexity.

On the other hand, [7, Theorem 1.1] provides sharp estimates for pseudometrics applied to tangent vectors in any direction, in terms of the Levi form.

- Our results about the relation between the local rank of the Levi form and the growth of the Bergman kernel (Proposition 5, Theorem 6) require \mathcal{C}^∞ smoothness in one direction, and with our method there is no way to bound the degree of smoothness required, so Fu’s method, requiring implicitly only \mathcal{C}^3 smoothness, yields a stronger result there.

2. PROOF OF THE SUFFICIENCY IN THEOREM 1

Let ρ be a \mathcal{C}^2 -smooth defining function of Ω near p . Suppose that there is some point $q \in \partial D$ near p such that the Levi form of ρ at q (restricted on $T_q^{\mathbb{C}}\partial\Omega$) has a non-zero eigenvalue.

If it is a negative eigenvalue, it follows by [11, Theorem 1.1] that

$$\limsup_{x \rightarrow 0+} \delta_\Omega^{3/4}(q_x) \kappa_\Omega(q_x; n_q) < \infty,$$

where $q_x = q + xn_q$. Therefore the left hand-side estimate in (1) cannot hold for normal vectors.

On the other hand, if the Levi form admits a positive eigenvalue, Proposition 3 shows that the right-hand side of (1) cannot hold for all tangential vectors.

Proposition 3. *Suppose that the Levi form admits a positive eigenvalue at $q \in \partial D$. Then there is an $X \in T_q^{\mathbb{C}} \partial \Omega$ such that*

$$\limsup_{x \rightarrow 0+} \kappa(q_x; X) = \infty.$$

In the case where all the eigenvalues are positive, stronger growth estimates are known, see e.g. [6, Chapter III, Theorem 3.1.1].

Proof. Using a translation, a rotation, and the implicit function theorem, we may always assume that $q = 0$, and that $\Omega = \{\rho < 0\}$, where near 0,

$$\rho(z) = \operatorname{Re} z_1 + O((\operatorname{Im} z_1)^2 + |z'|^2), \text{ where } z' = (z_2, \dots, z_n).$$

A further rotation lets us assume that the positive eigenvalue is in the z_2 -direction, and a dilation that it is equal to 1. Therefore

$$\rho(z) = \operatorname{Re} z_1 + |z_2|^2 + O((\operatorname{Im} z_1)^2 + |z_2||z'| + |z'|^2), \text{ where } z' = (z_3, \dots, z_n).$$

Then

$$\rho(z) \geq \operatorname{Re} z_1 + \frac{1}{2}|z_2|^2 - C((\operatorname{Im} z_1)^2 + |z'|^2),$$

and since passing to a smaller defining function, thus to a larger domain, can only decrease the Kobayashi metric, we may assume that ρ has this expression. At the cost of further dilations in z_1, z_2 and z' , we finally reduce ourselves to

$$\rho(z) = \operatorname{Re} z_1 + |z_2|^2 - ((\operatorname{Im} z_1)^2 + |z'|^2).$$

We estimate $\kappa_{\Omega}(z_{\delta}; X)$ where $z_{\delta} = (-\delta, 0, \dots, 0)$ ($\delta > 0$ small enough) and $X = (0, 1, 0, \dots, 0)$. Let φ be a holomorphic map from \mathbb{D} to Ω such that $\varphi(0) = z_{\delta}$ and $\varphi'(0) = X$. We will use with no further mention the fact that the Taylor coefficients of φ are bounded since Ω is. We have

$$\varphi(\zeta) = (-\delta, \lambda\zeta, 0, \dots, 0) + \zeta^2\psi(\zeta),$$

where $\psi(\zeta) = (\psi_1(\zeta), \dots, \psi_n(\zeta))$ is bounded for $|\zeta| \leq \frac{1}{2}$ (say). Denote $\tilde{\psi}(\zeta) = \frac{1}{\zeta}(\psi_1(\zeta) - \psi_1(0))$, which is also bounded. Then

$$\begin{aligned} \rho \circ \varphi(\zeta) &= -\delta + \operatorname{Re}(\psi_1(0)\zeta^2) + \operatorname{Re}(\tilde{\psi}_1(\zeta)\zeta^3) \\ &\quad + |\lambda|^2|\zeta|^2|1 + \zeta\psi_2(\zeta)|^2 - (\operatorname{Im}(\psi_1(\zeta)\zeta^2))^2 - |\zeta|^4 \sum_{j=3}^n |\psi_j(\zeta)|^2. \end{aligned}$$

Choose $\zeta = re^{i\theta}$ with $\operatorname{Re}(\psi_1(0)e^{2i\theta}) \geq 0$. Then since $\varphi(\mathbb{D}) \subset \Omega$, we have for $0 < r < 1$,

$$0 > -\delta + |\lambda|^2 r^2 - C_1 r^3 - C_2 r^4 \geq -\delta + |\lambda|^2 r^2 - C_3 r^3.$$

Choose $r = \delta^{1/3}$, we find $|\lambda| \leq (1 + C_3)^{1/2} \delta^{1/6}$, which means that $\kappa_\Omega(z_\delta; X) \gtrsim \delta^{-1/6}$, and therefore goes to infinity as δ goes to 0. \square

3. PROOF OF THE SUFFICIENCY IN THEOREM 2

We prove in this section that each of the conditions (2) or (3) implies that $\partial\Omega$ admits a Levi flat portion in a neighborhood of p by proving that if $\partial\Omega$ is not Levi flat in any neighborhood of p , then those estimates must fail.

Suppose $\partial\Omega$ is not Levi flat in U . Recall that the Levi form $\mathcal{L}\rho(q)$ is a semidefinite positive Hermitian form on $T_q^{\mathbb{C}}\partial\Omega \simeq \mathbb{C}^{n-1}$, and that if its rank $l_\Omega(q)$ is equal to l and $\mathbb{C}^{n-1} = T_1 \oplus T_2$ with $\dim T_1 = l$, $\mathcal{L}\rho(q)|_{T_1}$ is definite positive, and $\mathcal{L}\rho(q)|_{T_2} \equiv 0$.

Let now $k = \max_{q \in U \cap \partial\Omega} l_\Omega(q)$. Since the rank is a lower semicontinuous function, there exists a non empty open set V_1 such that $k = l_\Omega(q)$ for any $q \in V_1$. We choose such a q , and take coordinates so that q becomes the origin and $\rho(z) = \operatorname{Re} z_1 + \rho_2(z)$ with $\rho_2(z) = O(\|z\|^2)$, thus $T_q^{\mathbb{C}}\partial\Omega = \{0\} \times \mathbb{C}^{n-1}$.

Furthermore we choose coordinates on \mathbb{C}^{n-1} such that $\mathcal{L}\rho(q)|_{\{0\} \times \mathbb{C}^{n-1-k}} \equiv 0$ and $\mathcal{L}\rho(q)|_{\mathbb{C}^k \times \{0\}}$ is definite positive. This latter property is stable, more precisely there is a ball about the origin $V_2 \subset V_1$ such that $(\partial\Omega) \cap (\mathbb{C}^{k+1} \times \{0\}) \cap V_2$ is a strict pseudoconvex boundary in \mathbb{C}^{k+1} . Let $\Omega' = \{z' \in \mathbb{C}^{k+1} : (z', 0) \in \Omega\}$ (we will use the $'$ notation freely to denote the first $k+1$ coordinates in what follows). It is a pseudoconvex domain, a smoothing of $V_2 \cap \Omega'$ will be strictly pseudoconvex, and there exists $V_3 \subset V_2$ such that for $z = (z', 0) \in (\Omega' \times \{0\}) \cap V_3$, then $\delta_{\Omega'}(z') \asymp \delta_\Omega(z)$.

We now recall briefly how the Bergman kernel is estimated in strictly pseudoconvex domains. Given $z' \in \Omega'$, there exists a polydisk $P'_{z'} \Subset \Omega'$ with radii $c\delta_{\Omega'}(z')$ in the complex normal direction and $c\delta_{\Omega'}^{1/2}(z')$ in the complex tangential directions. It follows by [10, Theorem 3.5.1] (see also [4]) that

$$(4) \quad k_{\Omega'}(z') \asymp \delta_{\Omega'}(z')^{-1-k/2} \asymp \lambda_{2k+2}(P'_{z'})^{-1/2},$$

where λ_m stands for the Lebesgue measure in real dimension m .

Lemma 4. *There exist concentric balls about q , $V_5 \Subset V_4$ such that for any $z' \in V'_5$,*

$$k_{\Omega' \cap V'_4}(z') \asymp k_{\Omega \cap V_4}(z', 0).$$

By the localization property of the Bergman kernel [5, Proposition 1],

$$k_{\Omega'}(z') \asymp k_{\Omega}(z', 0).$$

Assuming Lemma 4, we prove that (2) must fail. Given any neighborhood U of p , there is $q \in U$ to which we can apply the Lemma and for $z = q + xn_q$, $x > 0$, (4) implies that

$$(5) \quad \delta_{\Omega}(z)k_{\Omega}(z) \asymp x^{-k/2} \rightarrow \infty \text{ as } x \rightarrow 0.$$

We turn to the failure of (3). For a given p and any neighborhood U , we choose $q \in U$ as at the beginning of this section. Then there exists a vector $X \in T_q^{\mathbb{C}}\partial\Omega$ such that

$$\liminf_{x \rightarrow 0^+} x^{1/2} \beta_{\Omega}(q + xn_q) > 0.$$

Indeed, with the coordinates chosen above, let $X = (X', 0)$ where $\|X'\| = 1$ and $X' \in \{0\} \times \mathbb{C}^k$. Since X' is a complex-tangential vector for the strictly pseudoconvex domain $\Omega \cap V_4$, it is known that $\beta_{\Omega'}(z', X') \asymp \delta_{\Omega'}(z')^{-1/2} \|X'\|$ (see [4]).

The Ohsawa–Takegoshi extension theorem [14] applied to the linear subspace \mathbb{C}^{k+1} and the domain Ω , implies that $m_{\Omega'}(z', X') \preceq m_{\Omega}((z', 0), (X', 0))$. In our new coordinates, $z = q + xn_q = (-x, 0, \dots, 0)$, $\delta_{\Omega}(z) = \delta_{\Omega'}(z') = x$. Then

$$\begin{aligned} \beta_{\Omega}(z, X) &= \frac{m_{\Omega}(z, X)}{k_{\Omega}(z)} \succeq \frac{m_{\Omega'}(z', X')}{k_{\Omega'}(z')} \\ &= \beta_{\Omega'}(z', X') \asymp \delta_{\Omega'}(z')^{-1/2} = \delta_{\Omega}(z)^{-1/2}. \end{aligned}$$

Proof of Lemma 4. The inequality $k_{\Omega' \cap V_4'}(z') \preceq k_{\Omega \cap V_4}(z', 0)$ follows easily from the Ohsawa–Takegoshi extension theorem, as above.

To prove the converse inequality, we first invoke a result of Sommer [15] and Kraut [12] that yields a foliation of a full neighborhood of q by a $2k+2$ -parameter family of complex manifolds F_{θ} of complex dimension $n-1-k$ such that $\rho|_{F_{\theta}}$ is constant for each θ . Some comments are in order : Sommer proved the theorem only in the case where ρ is \mathcal{C}^4 -smooth, Kraut gave a new proof that is valid if ρ is \mathcal{C}^2 -smooth. Both of them state the result only as a foliation of the hypersurface $\{\rho = 0\}$, but prove it for a whole neighborhood (and in fact need this in order to carry out their proofs). See [12, p. 310]: “Da diese charakteristischen Mannigfaltigkeiten notwendig auf den Flächen $f = \text{const.}$ verlaufen, blättern sie also diese komplex-analytisch.”

We need to have holomorphic parametrizations of the leaves depending continuously on θ . The leaves are obtained as integral manifolds of an integrable distribution of vector fields, whose value at each $q \in U$ are

vectors in the kernel of $\mathcal{L}\rho(q)$. Since ρ is \mathcal{C}^2 -smooth, those vectors can be chosen to depend continuously on q . The exponential maps of those vector fields are analytic (because the vector fields are holomorphic vector fields), and so not only their values, but also their derivatives with respect to the parameters of each leaf, depend continuously on θ .

Denote by V_6 a ball about q small enough so that the kernel of $\mathcal{L}\rho(\zeta)$ remains transverse to $\mathbb{C}^{k+1} \times \{0\}$ with a uniformly bounded angle for $\zeta \in V_6$, and therefore so does $F_\theta \cap V_6$ for each relevant θ . We then may parametrize those manifolds by $F_\theta \cap \mathbb{C}^{k+1} \times \{0\} = \{(\theta, 0)\}$, provided V_6 is chosen small enough.

Since the leaves depend continuously on θ , we can choose holomorphic maps $\Phi_\theta : G_\theta \rightarrow F_\theta \subset \mathbb{C}^n$, where G_θ is a neighborhood of the origin in \mathbb{C}^{n-1-k} such that the unit ball $\mathbb{B}^{n-1-k} \subset G_\theta$, for any $\theta \in V'_4$, where $V_4 \subset V_6$ is a ball about the origin; and furthermore, reducing V_4 as needed, for $\theta \in V_4$, $\Phi_\theta(G_\theta) \subset V_6$ and the Gram determinant of the image of the standard basis of \mathbb{C}^{n-1-k} by $D\Phi_\theta$ is bounded above and below uniformly in $\theta \in V'_4$. For some ball $V_5 \Subset V_4$, we may also assume that $P'_{z'} \subset V_4$ for any $z' \in V'_5$.

For any $f \in L^2(\Omega \cap V_4) \cap \mathcal{O}(\Omega \cap V_4)$, $z' \in V'_4$, then $f \circ \Phi_{z'}$ is holomorphic in a neighborhood of \mathbb{B}^{n-1-k} , so

$$|f(z')|^2 = |f \circ \Phi_{z'}(0)|^2 \leq c_1 \int_{\mathbb{B}^{n-1-k}} |f \circ \Phi_{z'}(\zeta)|^2 d\lambda_{2(n-1-k)}(\zeta),$$

where c_1 depends only on the dimension. For $z' \in V'_5$,

$$\begin{aligned} & \int_{P'_{z'}} |f(\xi)|^2 d\lambda_{2(k+1)}(\xi) \\ & \leq c_1 \int_{P'_{z'}} \int_{\mathbb{B}^{n-1-k}} |f \circ \Phi_\xi(\zeta)|^2 d\lambda_{2(n-1-k)}(\zeta) d\lambda_{2(k+1)}(\xi) \\ & \leq c_2 \int_{\tilde{P}_{(z',0)}} |f(\zeta)|^2 d\lambda_{2n}(\zeta) \leq c_2 \int_{\Omega \cap V_4} |f(\zeta)|^2 d\lambda_{2n}(\zeta), \end{aligned}$$

where $\tilde{P}_{(z',0)} = \bigcup \{\Phi_\xi(\mathbb{B}^{n-1-k}) : \xi \in P'_{z'}\}$, and the requirements about transversality and the differential of Φ_ξ ensure that the Jacobian determinant involved in the change of variables remains bounded.

Now if we are given a function f in the unit ball of $L^2(\Omega \cap V_4) \cap \mathcal{O}(\Omega \cap V_4)$, applying the mean value inequality as in (4) implies that $|f(z', 0)| \leq \lambda_{2k+2}(P'_{z'})^{-1/2} \asymp k_{\Omega' \cap V_4}(z')$. \square

4. THE RANK OF THE LEVI FORM

Observe that (5) proves a bit more than the failure of (2) when the boundary is not Levi flat. In particular, it still holds when $k = 0$ (Levi flat case), and gives an estimate of the growth of the Bergman kernel in terms of the rank of the Levi form. Thus we have the following corollary of the proof in section 3.

Proposition 5. *Let p be a boundary point of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\partial\Omega$ is \mathcal{C}^2 -smooth near p .*

If there exist a neighborhood U of p and constants $c > 1$, $m > 0$ such that for any $z \in \Omega \cap U$,

$$(6) \quad c^{-1} < k_{\Omega}(z)\delta_{\Omega}^m(z) < c,$$

then $2(m-1) \in \mathbb{N} \cup \{0\}$ and there exists a non empty open set $U_1 \subset U$ such that the Levi form of $\partial\Omega$ has constant rank $2(m-1)$ in U_1 .

Conversely, if there exists a neighborhood U of p such that the Levi form of $\partial\Omega$ has constant rank $2(m-1)$ in U , then there is a neighborhood $U_1 \subset U$ of p such that (6) holds for any $z \in \Omega \cap U_1$.

As a consequence, denoting by $l_{\Omega}(p)$ the rank of the Levi form of $\partial\Omega$ at p ,

$$\limsup_{\partial\Omega \ni q \rightarrow p} l_{\Omega}(q) = 2 \limsup_{\Omega \ni z \rightarrow p} \frac{\log k_{\Omega}(z)}{\log 1/\delta_{\Omega}(z)} - 1.$$

More generally, one may conjecture that (6) implies that the Levi form of $\partial\Omega$ has constant rank $2(m-1)$ near p . It is not difficult to see this (by dilatation of the coordinates) if the rank is maximal (i.e. $n-1$). On the other hand, (5) implies the conjecture when the rank is minimal (i.e. 0).

In general, it is difficult to say what happens to the foliation in complex manifolds near a degeneracy point, where the rank of the Levi form verifies $l_{\Omega}(p) < \limsup_{q \rightarrow p, q \neq p} l_{\Omega}(q)$. However, in the smooth case, we may confirm the above conjecture with the aid of the Catlin multitype [2].

Theorem 6. *Let p be a boundary point of a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\partial\Omega$ is \mathcal{C}^{∞} -smooth near p . Then if (6) holds in a neighborhood of p , we have $l_{\Omega}(p) = 2(m-1)$.*

Proof. In view of Proposition 5, it is enough to consider the case where the rank is not locally constant at p . By lower semi continuity, this means that there is k , $1 \leq k \leq n-1$, such that $\limsup_{q \rightarrow p, q \neq p} l_{\Omega}(q) = k$ and that $l_{\Omega}(p) \leq k-1$. Since we assume (6), $k = 2(m-1)$.

We refer the reader to [2] for a complete definition of the Catlin multitype of Ω at p . Here we will only recall that it is an n -tuple

$M = (1, m_2, \dots, m_n)$, $2 \leq m_2 \leq \dots \leq m_n$, with the following property: if $\Lambda = (\lambda_1, \dots, \lambda_n) \in [1, \infty]^n$ is a weight such that $\Lambda < M$ in the lexicographical order, then there exists another weight Λ' with $\Lambda < \Lambda' \leq M$ and a defining function ρ for Ω and a system of complex coordinates which is distinguished with respect to Λ' , i.e.

$$\text{if } \alpha, \beta \in \mathbb{N}^n \text{ verify } \sum_{j=1}^n \frac{\alpha_j + \beta_j}{\lambda'_j} < 1, \text{ then } \frac{\partial^{|\alpha|+|\beta|} \rho}{\partial z^\alpha \partial \bar{z}^\beta}(p) = 0.$$

Catlin proved that the multitype is upper semicontinuous with respect to lexicographical order [2, Theorem 1 (1)].

It is well known that the multitype of a strictly pseudoconvex point is $(1, 2, \dots, 2)$. If $l_\Omega \equiv k$ in a neighborhood of a point q , we can take holomorphic coordinates (z_1, z_2, \dots, z_n) such that $\{z_{k+2} = \dots = z_n = 0\}$ represents the $n - 1 - k$ complex dimensional manifold contained in $\partial\Omega$ and passing through q which exists by Sommer's theorem [15]. Then all the derivatives of ρ vanish in those directions, while the complex tangential directions z_2, \dots, z_{k+1} are "strictly pseudoconvex" directions and z_1 is the complex normal direction. Therefore the type will be given by $m_j = 2$, $2 \leq j \leq k + 1$, and $m_j = \infty$, $k + 2 \leq j \leq n$.

Our assumption on p and the upper semi continuity of the multitype now imply that at p , $m_j \geq 2$, $2 \leq j \leq k + 1$, $m_j = \infty$, $k + 2 \leq j \leq n$, and $m_{k+1} > 2$ (otherwise $l_\Omega(p) = k$).

Lemma 7. *Under the above assumption on the multitype, for any r large enough and any $\varepsilon > 0$, there exists $A_r > 0$, holomorphic coordinates z_1, \dots, z_n and U a neighborhood of 0 such that for any $z \in U$,*

$$\rho(z) \leq \operatorname{Re} z_1 + \varepsilon |z_1| + A_r \left(\sum_{j=2}^k |z_j|^2 + |z_{k+1}|^3 + \sum_{j=k+2}^n |z_j|^r \right).$$

Accepting Lemma 7, we see that for $c > 0$ an appropriate constant and $x > 0$ small enough, there is a polydisk contained in Ω centered at $p + xn_p = (-x, 0, \dots, 0)$ with respective radii cx in the z_1 direction, $cx^{1/2}$ in the z_2, \dots, z_k directions, $cx^{1/3}$ in the z_{k+1} direction, and $cx^{1/r}$ in the z_{k+2}, \dots, z_n directions. The usual volume estimate yields that

$$\begin{aligned} \log k_\Omega(p + xn_p) &\leq \left(1 + \frac{k-1}{2} + \frac{1}{3} + \frac{n-1-k}{r} \right) \log \frac{1}{x} + O(1) \\ &\leq \left(\frac{5}{6} + \frac{k}{2} + \frac{n-1-k}{r} \right) \log \frac{1}{x} + O(1) \ll \left(1 + \frac{k}{2} \right) \log \frac{1}{x} \end{aligned}$$

if r is chosen large enough. This contradicts (6) with $m = 1 + \frac{k}{2}$, which was our assumption, so in fact $l_\Omega(p) = k$. \square

Proof of Lemma 7. Pick an integer $r \geq 3$ such that $\frac{1}{m_{k+1}} + \frac{1}{2r} < \frac{1}{2}$.

By the assumption on multitype, there exists an admissible weight $\Lambda = (1, \lambda_2, \dots, \lambda_n)$ with $\lambda_j \geq m_j \geq 2$, $2 \leq j \leq k$, $\lambda_{k+1} \geq m_{k+1} > 2$, $\lambda_j \geq r^2$, $k+2 \leq j \leq n$. There is an admissible system of coordinates in which we can write the Taylor formula up to order r :

$$\rho(z) = \operatorname{Re} z_1 + \sum_{2 \leq |\alpha| + |\beta| \leq r} \frac{1}{\alpha! \beta!} \frac{\partial^{|\alpha| + |\beta|} \rho}{\partial z^\alpha \partial \bar{z}^\beta}(0) z^\alpha \bar{z}^\beta + o(|z|^r).$$

The remainder term will be dominated by the desired estimates. We now estimate the terms in the sum. For any nonzero term, we must have $\sum_{j=1}^n \frac{\alpha_j + \beta_j}{\lambda_j} \geq 1$.

If $\alpha_1 + \beta_1 \geq 1$, then the corresponding term is an $O(|z_1||z|)$, so will be bound by $\varepsilon|z_1|$. From now on assume $\alpha_1 + \beta_1 = 0$.

If $\sum_{j=2}^k \alpha_j + \beta_j \geq 2$, then the corresponding term is an $O(\sum_{j=2}^k |z_j|^2)$. Write $\alpha' = (\alpha_1, \dots, \alpha_k)$, $\beta' = (\beta_1, \dots, \beta_k)$, $\alpha'' = (\alpha_{k+1}, \dots, \alpha_n)$, $\beta'' = (\beta_{k+1}, \dots, \beta_n)$.

If $\sum_{j=2}^k \alpha_j + \beta_j = 1$, then

$$\left| z^{\alpha'} \bar{z}^{\beta'} z^{\alpha''} \bar{z}^{\beta''} \right| = O \left(\left| z^{2\alpha'} \bar{z}^{2\beta'} \right| + \left| z^{2\alpha''} \bar{z}^{2\beta''} \right| \right),$$

so the first term in that last sum is an $O(\sum_{j=2}^k |z_j|^2)$, and the second one has exponents which verify

$$\sum_{j=k+1}^n \frac{2\alpha_j + 2\beta_j}{\lambda_j} \geq 2 \left(1 - \frac{1}{m_2}\right) \geq 1,$$

so will be treated as the next case.

If $\sum_{j=2}^k \alpha_j + \beta_j = 0$, then either $\alpha_{k+1} + \beta_{k+1} \geq 3$ and the corresponding term is an $O(|z_{k+1}|^3)$. Otherwise,

$$\frac{1}{r^2} \sum_{j=k+2}^n \alpha_j + \beta_j \geq \sum_{j=k+2}^n \frac{\alpha_j + \beta_j}{\lambda_j} \geq 1 - \frac{2}{\lambda_{k+1}} > \frac{1}{r},$$

so $\sum_{j=k+2}^n \alpha_j + \beta_j \geq r$ and the corresponding term is an $O(\sum_{j=k+2}^k |z_j|^r)$.
□

5. PROOF OF THE ESTIMATES

The key point will be the following:

Lemma 8. *Suppose that $\partial\Omega$ is Levi-flat near $p \in \partial\Omega$. Then there exist neighborhoods $V \Subset U$ of p such that for any $q \in \partial\Omega \cap V$ one may find a biholomorphism Φ_q defined on U such that $\Phi_q(q) = 0$, and $\Phi_q(\Omega \cap U) = \{w \in U : \rho_q(w) < 0\}$, where ρ_q is a \mathcal{C}^2 -smooth function*

such that $\rho_q(w) = \operatorname{Re} w_1 - f_q(w)$, $\operatorname{ord}_0 f_q \geq 2$ and $f_q(0, w'') = 0$, where $w = (w_1, w'')$. Moreover, the Jacobian determinant of Φ_q is identically 1, the C^2 -norms of Φ_q and f_q are bounded on U , uniformly in q , and $\|\Phi_q(z) + q - z\| \leq C\|z'' - q''\|$.

Notice that this is not the same Φ as in section 3. The proof in this section may seem similar to that in section 3, in that it also depends on a local foliation; the difference being that now the leaves of the foliation have the maximum complex dimension $n - 1$, and so there is no vector in the complex tangent space to $\partial\Omega$ that is transverse to that foliation (as was the case in section 3).

Proof. We may assume that $p = 0$ and that a defining function of Ω near 0 is given by

$$\rho(z) = \operatorname{Re} z_1 - g(\operatorname{Im} z_1, z''), \quad \operatorname{ord}_0 g \geq 2.$$

We know from [1] that $\partial\Omega$ near 0 is foliated by complex manifolds of the form

$$(\varphi(y, z''), z''), \quad (y, z'') \in (-\varepsilon, \varepsilon) \times \varepsilon\mathbb{D}^{n-1},$$

where $\varepsilon > 0$, φ is a \mathcal{C}^2 -smooth function, $\varphi(y, \cdot) \in \mathcal{O}(\varepsilon\mathbb{D}^{n-1})$ and $\varphi(y, 0) = g(y, 0) + iy$. The Implicit Function Theorem implies that there is a \mathcal{C}^2 -smooth function $y(q)$ such that $q = (\varphi(y(q), q''), q'')$.

The following map will be the desired biholomorphism if we choose the neighborhoods in an appropriate way:

$$\Phi_q(z_1, z'') = (z_1 - \varphi(y(q), z''), z'' - q'').$$

Indeed, let \mathcal{L}_q denote the unique leaf in the foliation passing through the point q . Then $\Phi_q(\mathcal{L}_q) = \{w_1 = 0\}$, so $\{w_1 = 0\} \subset \partial\Phi_q(\Omega)$ (near 0) and we may set $\rho_q = \rho \circ \Phi_q^{-1}$. \square

Now, we are ready to prove the estimates in Theorem 1 under the respective conditions. It follows from the above lemma that there exist neighborhoods $V \Subset U$ of p and a constant $\varepsilon > 0$ such that for any point $z \in D \cap V$ one has that

$$\begin{aligned} \{w : \operatorname{Re} w_1 + \varepsilon|w_1| < 0\} \cap (\varepsilon^2\mathbb{D}^n) &=: F \subset \Phi_{p(z)}(\Omega \cap U) \subset \\ G &:= \{w : \operatorname{Re} w_1 - \varepsilon|w_1| < 0\} \cap (2\varepsilon^2\mathbb{D}^n). \end{aligned}$$

Note that if $w(z) = \Phi_{p(z)}(z)$, then $\|w(z) + (\delta_\Omega(z), 0'')\|/\delta_\Omega(z) \rightarrow 0$ as $z \rightarrow 0$.

Moreover, if $Y(z) = (\Phi_{p(z)})_{*,z}(X)$, then $Y_1(z) = (1 + O(\|z\|))X_N$ and $Y''(z) = X''$. Using this and, for example, the product property of the

Kobayashi metric ($\kappa_{D_1 \times D_2} = \max(\kappa_{D_1}, \kappa_{D_2})$; cf. [9]) and a dilatation of the coordinates, one may find a constant $c_2 > 0$ such that

$$\begin{aligned} \|X_N\|/(c_2\delta_\Omega(z)) &\leq \kappa_G(w(z); Y(z)) \leq \kappa_{\Omega \cap U}(z; X) \leq \\ &\kappa_F(w(z); Y(z)) \leq c_2\|X_N\|/\delta_\Omega(z) + c_2\|X\|. \end{aligned}$$

To get (1), it remains to use that $\kappa_\Omega \leq \kappa_{\Omega \cap U} \leq c_3\kappa_\Omega$ (cf. [9]) and the fact that $\kappa_\Omega(z; X) \geq \|X\|/\text{diam}(\Omega)$.

The proof of the estimate (3) is similar. Indeed, since $m_{D_2} \leq m_{D_1}$ and $k_{D_2} \leq k_{D_1}$ if $D_1 \subset D_2$, then

$$\beta_G(w(z); Y(z)/s(w(z))) \leq \beta_{\Omega \cap U}(z; X) \leq \beta_F(w(z); Y(z))s(w(z)),$$

where $s = k_F/k_G$. Using, for example, the product property of the Bergman metric ($\beta_{D_1 \times D_2}^2 = \beta_{D_1}^2 + \beta_{D_2}^2$; cf. [9]) and a dilatation of the coordinates, one may find a constant $c_4 > 0$ such that

$$\begin{aligned} \|X_N\|/c_2\delta_\Omega(z) &\leq \beta_G(w(z); Y(z)) \leq \beta_F(w(z); Y(z)) \leq \\ &c_4\|X_N\|/\delta_\Omega(z) + c_4\|Y(z)\|, \quad s(w(z)) \leq c_4. \end{aligned}$$

To complete the proof of the estimate (3), it remains to use that $c_5\beta_\Omega \leq \beta_{\Omega \cap U} \leq \beta_\Omega/c_5$ (see [5, Proposition 1]) and $\beta_\Omega(z; X) \geq \|X\|/\text{diam}(\Omega)$.

Finally, as explained at the beginning of section 4, the estimate (2) follows from (5) in the case where the rank $k = 0$. We could also use easier versions of the arguments above.

6. SHARP ESTIMATES

The estimate in Theorem 1 can be made sharp in the convex case.

Proposition 9. *Let $\Omega \subset \mathbb{C}^n$ be a domain with Levi-flat boundary near a \mathcal{C}^2 -smooth boundary point p . Assume in addition that Ω does not intersect the real tangent hyperplane to $\partial\Omega$ at any boundary point near p (for example, if Ω is convex). Denote by α_Ω the Carathéodory or the Kobayashi metric. Then there exists a neighborhood U_p of p and a constant $c_p > 0$ such that*

$$0 \leq \alpha_\Omega(z; X) - \|X_N\|/2\delta_\Omega(z) \leq c_p\|X\|$$

for any $z \in \Omega \cap U_p$ and any $X \in \mathbb{C}^n$.

Recall that the Carathéodory metric is defined by

$$\gamma_\Omega(z; X) = \sup\{|f'(z)X| : f \in \mathcal{O}(D, \mathbb{D})\}.$$

Proof. For the lower bound observe that Ω is on the one side Π_{p_z} of the real tangent hyperplane plane to $\partial\Omega$ at p_z and hence

$$\gamma_\Omega(z; X) \geq \gamma_{\Pi_{p_z}}(z; X) \geq \|X_N\|/2\delta_\Omega(z).$$

For the upper estimate choose a neighborhood V_p such that $\Omega_p = \Omega \cap V_p$ is convex. Then $\kappa_\Omega \leq \kappa_{\Omega_p} = \gamma_{\Omega_p}$ and hence

$$\kappa_\Omega(z; X) \leq \kappa_{\Omega_p}(z; X_N) + \kappa_{\Omega_p}(z; X_T).$$

We already know the the Levi flatness implies that

$$\limsup_{z \rightarrow p} \kappa_{\Omega_p}(z; X_T) \leq c' \|X_T\|.$$

On the other hand, the proof of Proposition 10 implies that

$$\limsup_{z \rightarrow p} (\kappa_{\Omega_p}(z; X_N) - \|X_N\|/2\delta_\Omega(z)) \leq c'' \|X_N\|$$

which completes the proof. \square

Finally, we present a sharp estimate for invariant metrics in the planar case.

Proposition 10. *Let p be a $C^{1,1}$ -smooth boundary point of a planar domain D . Then*

$$(7) \quad \limsup_{D \ni z \rightarrow p} |\alpha_D(z) - 1/2\delta_D(z)| < \infty,$$

$$(8) \quad \limsup_{D \ni z \rightarrow p} |\beta_D(z) - 1/\sqrt{2}\delta_D(z)| < \infty,$$

where α is the Carathéodory or Kobayashi metric, and β_D is the Bergman metric, all taken at the unit vector. Moreover,

$$(9) \quad \limsup_{z \rightarrow p} |k_D(z) - 1/2\sqrt{\pi}\delta_D(z)| < \infty.$$

In the C^1 -smooth case a weaker result is known, namely (see [8], Proposition 2 and the remark in the end)

$$\lim_{z \rightarrow p} \alpha_D(z)\delta_D(z) = 1/2, \quad \lim_{z \rightarrow p} \beta_D(z)\delta_D(z) = 1/\sqrt{2}.$$

Proof. Let r be the double signed distance to ∂D . For $\zeta \in D$ near p , set $\Phi_\zeta(z) = \partial r / \partial z(p_\zeta)(z - p_\zeta)$, $D_\zeta = \Phi_\zeta(D)$ and $\delta_\zeta = \Phi_\zeta(\zeta)$. Note that

$$\alpha_D(z) = \alpha_{D_\zeta}(\eta_\zeta), \quad \delta_D(z) = \eta_\zeta.$$

Then there exists an $\varepsilon > 0$ such that

$$\{|z - \varepsilon| < \varepsilon\} =: F_\varepsilon \subset D_\zeta \subset G_\varepsilon := \{|z + \varepsilon| > \varepsilon\}$$

for any $\zeta \in D$ near p . Hence

$$\frac{1}{\eta_\zeta(2 - \eta_\zeta/\varepsilon)} = \kappa_{F_\varepsilon}(\eta) \geq \kappa_{D_\zeta}(\eta_\zeta) \geq \gamma_{D_\zeta}(\eta_\zeta) \geq \gamma_{G_\varepsilon}(\eta_\zeta) = \frac{1}{\eta_\zeta(2 + \eta_\zeta/\varepsilon)}$$

which implies (7).

The proof of (8) is similar. Let $m_\Omega(z) = \beta_\Omega(z)k_\Omega(z)$. Recall that $k_\Omega \leq k_\Theta$ and $m_\Omega \leq m_\Theta$ if $\Theta \subset \Omega$. Then

$$\frac{\sqrt{2}(2 + \eta_\zeta/\varepsilon)}{\eta_\zeta(2 - \eta_\zeta/\varepsilon)^2} = \frac{m_{F_\varepsilon}(\zeta)}{k_{G_\varepsilon}(\zeta)} \geq \beta_{D_\zeta}(\eta_\zeta) \geq \frac{m_{G_\varepsilon}(\eta_\zeta)}{k_{F_\varepsilon}(\eta_\zeta)} = \frac{\sqrt{2}(2 - \eta_\zeta/\varepsilon)}{\eta_\zeta(2 + \eta_\zeta/\varepsilon)^2}$$

which implies (8).

We skip the proof of (9), since it is even easier than that of (8). \square

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